1、对于1.6.5（12）题，求 $\int_{-1}^{1}\frac{x\,dx}{\sqrt{5-4x}}$

解法1：令 $t=5-4x$ $\quad \int_{-1}^{1}\frac{x\,dx}{\sqrt{5-4x}}=\int_{9}^{1}\frac{\frac{5-t}{4}\cdot\left(-\frac{1}{4}\right)}{\sqrt{t}}\,dt$

$$=-\int_{9}^{1}\left(\frac{5}{4}\cdot\frac{1}{\sqrt{t}}-\frac{1}{4}\cdot\sqrt{t}\right)\frac{1}{4}\,dt$$

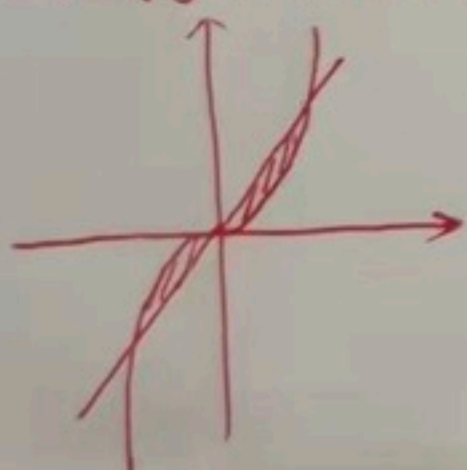
$$=-\frac{1}{4}\cdot\left(\frac{5}{2}\sqrt{t}-\frac{1}{6}\cdot t^{\frac{3}{2}}\right)\Big|_{9}^{1}=\frac{1}{6}$$

解法2：令 $t=\sqrt{5-4x}$ $\quad x=\frac{5-t^2}{4}$

$$\int_{-1}^{1}\frac{x\,dx}{\sqrt{5-4x}}=\int_{3}^{1}\frac{\frac{5-t^2}{4}}{t}\cdot\left(-\frac{t}{2}\right)dt=\left(-\frac{5}{8}t+\frac{1}{24}t^3\right)\Big|_{3}^{1}$$

$$=\frac{1}{6}$$

2. 对于1.7.1(11)，求 $y=x^3$ 与 $y=2x$ 围成的图形的面积



面积应是第一象限和第三象限组成，有的同学只求了一半。

3. 分部积分：$\int_a^b g(x)f(x)\,dx=\int_a^b f(x)\,dG(x)$ $\quad$ $G(x)$为$g(x)$的原函数

$$=G(x)f(x)\Big|_a^b-\int_a^b G(x)\,df(x)$$

有的同学将$G(x)$，也就是$g(x)$的原函数写成了$g(x)$的导函数，导致结果错误。

1.6.1 计算下列函数的定积分

(3) $\int_1^2 (x^2+\frac{1}{x^4})dx$

$=\int_1^2 (x^2+x^{-4})dx$

$=\int_1^2 \left((\frac{1}{3}x^3)'+(-\frac{1}{3}x^{-3})'\right)dx$

$=(\frac{1}{3}x^3-\frac{1}{3}x^{-3})\Big|_1^2$

$=\frac{1}{3}\cdot 8-\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{3}\cdot 1+\frac{1}{3}\cdot 1$

$=\frac{21}{8}$

(9) $\int_0^3 |x-1|dx$

$=\int_0^1 |x-1|dx+\int_1^2 |x-1|dx$

$=\int_0^1 (1-x)dx+\int_1^2 (x-1)dx$

$=(x-\frac{1}{2}x^2)\Big|_0^1+(\frac{1}{2}x^2-x)\Big|_1^3$

$=1-\frac{1}{2}-0+\frac{1}{2}\cdot 9-3-\frac{1}{2}\cdot 1+1$

$=\frac{5}{2}$

(12) $\int_{-1}^1 f(x)dx \qquad f(x)=\begin{cases}x^2, & x\geq 0\\ x, & x\leq 0\end{cases}$

$=\int_{-1}^0 f(x)dx+\int_0^1 f(x)dx$

$=\int_{-1}^0 x\,dx+\int_0^1 x^2dx$

$=\frac{1}{2}x^2\Big|_{-1}^0+\frac{1}{3}x^3\Big|_0^1$

$=0-\frac{1}{2}+\frac{1}{3}\cdot 1-0=-\frac{1}{6}$

1.6.3 设 $\int_0^1 (2x+k)dx=2$，求 k.

解: $\int_0^1 (2x+k)dx$

$=(x^2+kx)\Big|_0^1$

$=1+k-0$

$=2$

$\Rightarrow k=1$

1.6.5. 用换元法计算下列定积分.

(1) $\int_1^2 \frac{1}{2x-1} dx$

$\overset{2x-1=t}{=\!=} \int_1^2 \frac{1}{2x-1} dx \quad \begin{cases} x=2时, t=3 \\ x=1时, t=1 \end{cases}$

$= \int_1^3 \frac{1}{t} d\frac{t+1}{2}$

$= \int_1^3 \frac{1}{2t} dt$

$= \frac{1}{2} \ln t \Big|_1^3$

$= \frac{1}{2}\ln 3 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 3$

(2) $\int_0^{\frac{\pi}{2}} \cos^5 x \sin 2x \, dx$

$= \int_0^{\frac{\pi}{2}} \cos^5 x \cdot 2\sin x \cos x \, dx$

$= 2\int_0^{\frac{\pi}{2}} \cos^6 x \sin x \, dx$

$\overset{\cos x = t}{=\!=} 2\int_1^0 t^6 (-t)' dx$

$= 2\int_1^0 t^6 d(-t)$

$= -2\int_1^0 t^6 dt$

$= -2 \cdot \frac{1}{7} \cdot t^7 \Big|_1^0 = 0 + 2 \cdot \frac{1}{7} = \frac{2}{7}$

(7) $\int_0^3 x(1+x^2)^{\frac{1}{5}} dx$

$= \int_0^3 (1+x^2)^{\frac{1}{5}} d\frac{1}{2}x^2$

$= \int_0^3 \frac{1}{2}(1+x^2)^{\frac{1}{5}} d(1+x^2)$

$\overset{x^2+1=t}{=\!=} \int_1^{10} \frac{1}{2} t^{\frac{1}{5}} dt$

$= \frac{1}{2} \cdot \frac{5}{6} \cdot t^{\frac{6}{5}} \Big|_1^{10}$

$= \frac{5}{12} \cdot 10^{\frac{6}{5}} - \frac{5}{12}$

(10) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x-\cos^3 x}\,dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x(1-\cos^2 x)}\,dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x}\cdot|\sin x|\,dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\sin x\,dx - \int_{-\frac{\pi}{2}}^{0} \sqrt{\cos x}\sin x\,dx$ $\qquad \cos x = t$

$= \int_1^0 -\sqrt{t}\,dt - \int_0^1 -\sqrt{t}\,dt$

$= -\frac{2}{3}t^{\frac{3}{2}}\Big|_1^0 + \frac{2}{3}t^{\frac{3}{2}}\Big|_0^1$

$= \frac{4}{3}$

(12) $\int_{-1}^{1} \frac{x\,dx}{\sqrt{5-4x}}$

$\overset{t=5-4x}{=\!=\!=} \int_9^1 \frac{\frac{5-t}{4}}{\sqrt{t}}\, d\frac{5-t}{4} = \int_9^1 \left(\frac{5}{4}\cdot t^{-\frac{1}{2}} - \frac{1}{4}t^{\frac{1}{2}}\right)\cdot\left(-\frac{1}{4}\right)dt$

$= \left(-\frac{1}{4}\right)\left(\frac{5}{2}t^{\frac{1}{2}} - \frac{1}{6}t^{\frac{3}{2}}\right)\Big|_9^1$

$= \left(-\frac{1}{4}\right)\cdot\left(\frac{5}{2} - \frac{1}{6} - \left(\frac{5}{2}\cdot 3 - \frac{1}{6}\cdot 27\right)\right) = \frac{1}{4}\cdot\frac{2}{3} = \frac{1}{6}$

(14) $\int_0^1 \frac{x^{\frac{3}{2}}dx}{1+x}$

$\overset{x=t^2}{=\!=\!=} \int_0^1 \frac{t^3\,dt^2}{1+t^2} = \int_0^1 \frac{2t^4\,dt}{1+t^2} = 2\int_0^1 \frac{(t^2+1)(t^2-1)+1}{t^2+1}\,dt$

$= 2\int_0^1 (t^2-1)\,dt + 2\int_0^1 \frac{1}{t^2+1}\,dt$

$= 2\cdot\left(\frac{1}{3}t^3 - t\right)\Big|_0^1 + 2\arctan t\Big|_0^1$

$= 2\cdot\left(\frac{1}{3}-1\right) + 2\cdot\frac{\pi}{4} = \frac{\pi}{2} - \frac{4}{3}$

(15) $\int_{1}^{\sqrt{3}} \frac{dx}{x\sqrt{x^2+1}}$

$$\xlongequal{x=\tan t} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\tan t}{\tan t \cdot \frac{1}{\cos t}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\frac{1}{\cos^2 t}}{\frac{\sin t}{\cos t} \cdot \frac{1}{\cos t}} dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin t} dt$$

$$= \ln \frac{1-\cos t}{\sin t} \Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \ln \frac{\sqrt{3}}{3} - \ln(\sqrt{2}-1) \qquad \left(\ln \frac{1-\cos t}{\sin t}\right)' = \frac{1}{\sin t}$$

1.6.7. 用分部积分法计算下列定积分。

(4) $\int_{e^{-1}}^{e} |\ln x| dx$

$$= \int_{1}^{e} \ln x dx - \int_{e^{-1}}^{1} \ln x dx$$

$$\xlongequal{分部积分} x\ln x \Big|_{1}^{e} - \int_{1}^{e} x d\ln x - \left(x\ln x \Big|_{e^{-1}}^{1} - \int_{e^{-1}}^{1} x d\ln x \right)$$

$$= e - 0 - \int_{1}^{e} 1 \cdot dx - \left(e^{-1} - \int_{e^{-1}}^{1} 1 \cdot dx\right)$$

$$= e - (e-1) - (e^{-1} - (1-e^{-1}))$$

$$= 2 - 2e^{-1}$$

(5) $\int_0^1 x\arctan x\,dx$

分部积分 $\int_0^1 \arctan x\, d\frac{x^2}{2} = \frac{x^2}{2}\arctan x\Big|_0^1 - \int_0^1 \frac{x^2}{2}\, d\arctan x$

$= \frac{1}{2}\cdot\frac{\pi}{4} - \int_0^1 \frac{x^2}{2}\cdot\frac{1}{1+x^2}\,dx = \frac{\pi}{8} - \frac{1}{2}\int_0^1 \left(1-\frac{1}{1+x^2}\right)dx$

$= \frac{\pi}{8} - \frac{1}{2}\cdot(x-\arctan x)\Big|_0^1 = \frac{\pi}{8} - \frac{1}{2}\left(1-\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$

1.6.8 设 $f(2x+1) = xe^x$, 求 $\int_3^5 f(t)dt$

解: 设 $2x+1=t$　　$x = \frac{t-1}{2}$

$f(t) = \frac{t-1}{2}\cdot e^{\frac{t-1}{2}}$

$\int_3^5 f(t)\,dt = \int_3^5 \frac{t-1}{2}\cdot e^{\frac{t-1}{2}}\,dt$

$\overset{\frac{t-1}{2}=m}{=\!=} \int_1^2 m\cdot e^m\, d(2m+1)$

$= 2\int_1^2 m\cdot e^m\,dm = 2\int_1^2 m\,de^m = 2\cdot\left(me^m\Big|_1^2 - \int_1^2 e^m dm\right) = 2(2e^2-e-(e^2-e))$

$= 2e^2$

$\cancel{\overset{分部积分}{=\!=}\ 2\int_1^2 e^m\, d\frac{1}{2}m^2}$

$\cancel{= \int_1^2 e^m\,dm^2 = e^m\cdot m^2\Big|_1^2 - \int_1^2}$

1.6.9 用适当方法计算下列定积分

(1) $\int_{-1}^1 (x-\sqrt{1-x^2})^2 dx$

$= \int_{-1}^1 [x^2+(1-x^2)-2x\sqrt{1-x^2}]\,dx$

$= \int_{-1}^1 1-2x\sqrt{1-x^2}\,dx$

$= \int_{-1}^1 1\,dx - \int_{-1}^1 \sqrt{1-x^2}\,dx^2$

$= 2 + \int_{-1}^1 \sqrt{1-x^2}\, d(1-x^2)$　　令 $1-x^2=t$

$= 2 + \int_0^0 \sqrt{t}\,dt$

$= 2$

(2) $\int_0^1 \frac{dx}{e^x+e^{-x}}$

$= \int_0^1 \frac{e^x}{e^{2x}+1} dx$

$= \int_0^1 \frac{de^x}{(e^x)^2+1}$

$= \arctan e^x \Big|_0^1$

$= \arctan e - \arctan 1$

$= \arctan e - \frac{\pi}{4}$

(5) $\int_0^2 \frac{dx}{\sqrt{x+1}+\sqrt{(x+1)^3}}$ $\quad d\sqrt{x+1} = \frac{dx}{2\sqrt{x+1}}$

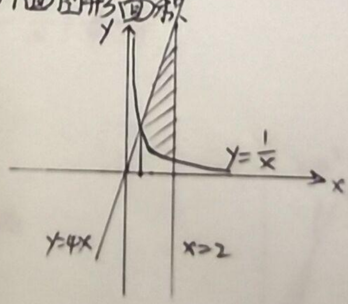
$= \int_0^2 \frac{dx}{\sqrt{x+1}\,(1+\sqrt{(x+1)^2})}$

$= \int_0^2 \frac{2d\sqrt{x+1}}{1+(\sqrt{x+1})^2}$

$\xlongequal{t=\sqrt{x+1}} \int_1^{\sqrt{3}} \frac{2dt}{1+t^2} = 2\arctan t \Big|_1^{\sqrt{3}} = 2\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{\pi}{6}$

1.7.1 求下列给定曲线围成的平面图形面积

(1) $y=\frac{1}{x}$, $y=4x$, $x=2$;



解: $y=\frac{1}{x}$ 与 $y=4x$ 的交点为 $(\frac{1}{2},2)$

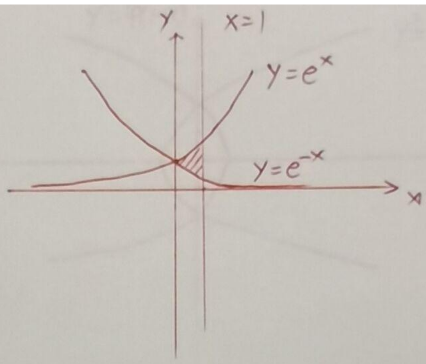
$S = \int_{\frac{1}{2}}^2 (4x-\frac{1}{x})\,dx$

$= (2x^2-\ln x)\Big|_{\frac{1}{2}}^2$

$= 8-\ln 2-(\frac{1}{2}-\ln\frac{1}{2})$

$= \frac{15}{2}-2\ln 2$

(4) $y=e^{x}$, $y=e^{-x}$, $x=1$;



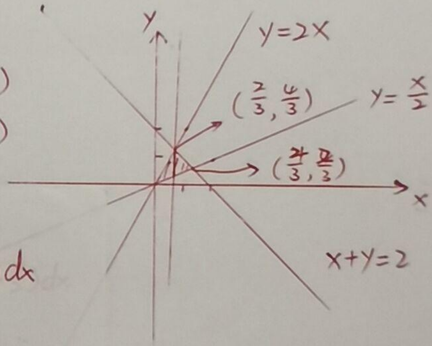
解: $S=\int_0^1 (e^{x}-e^{-x})dx$

$=(e^{x}+e^{-x})\big|_0^1$

$=e+e^{-1}-(1+1)$

$=e+\frac{1}{e}-2$

(6) $y=2x$, $y=\frac{x}{2}$, $x+y=2$;



解: $y=2x$与$x+y=2$的交点为$(\frac{2}{3},\frac{4}{3})$

$y=\frac{x}{2}$与$x+y=2$的交点为$(\frac{4}{3},\frac{2}{3})$

$S=S_1+S_2$

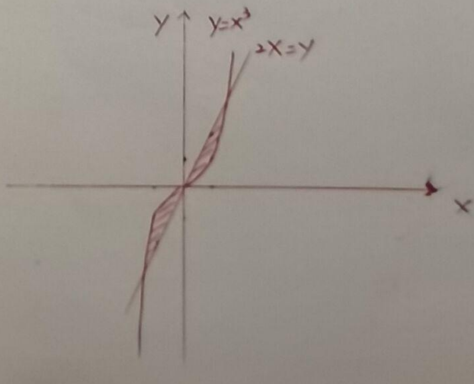
$=\int_0^{\frac{2}{3}}(2x-\frac{x}{2})dx+\int_{\frac{2}{3}}^{\frac{4}{3}}(2-x-\frac{x}{2})dx$

$=(x^2-\frac{1}{4}x^2)\big|_0^{\frac{2}{3}}+(2x-\frac{3}{4}x^2)\big|_{\frac{2}{3}}^{\frac{4}{3}}$

$=\frac{4}{9}-\frac{1}{4}\cdot\frac{4}{9}+\frac{8}{3}-\frac{3}{4}\cdot\frac{16}{9}-\frac{4}{3}+\frac{3}{4}\cdot\frac{4}{9}$

$=\frac{2}{3}$

(11) $y=x^3$与$y=2x$



解: $y=x^3$与$y=2x$的交点坐标为

$(0,0)$, $(\sqrt{2},2\sqrt{2})$, $(-\sqrt{2},-2\sqrt{2})$

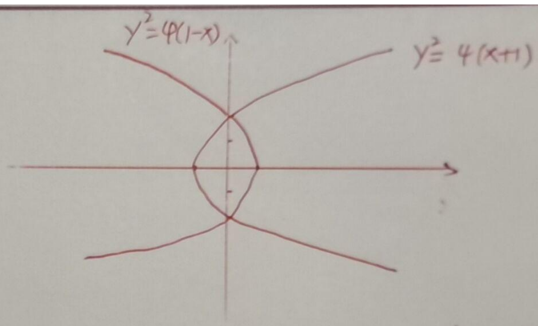
由对称性

$S=2\cdot\int_0^{\sqrt{2}}(2x-x^3)dx$

$=2\cdot(x^2-\frac{1}{4}x^4)\big|_0^{\sqrt{2}}$

$=2\cdot(2-\frac{1}{4}\cdot 4)=2$

(13) $y^2=4(x+1)$与$y^2=4(1-x)$



解: $y^2=4(x+1)$与$y^2=4(1-x)$

的交点为$(0,2)$, $(0,-2)$

$S=\int_{-2}^{2}\left[(1-\frac{y^2}{4})-(\frac{y^2}{4}-1)\right]dy$

$=\int_{-2}^{2}2-\frac{y^2}{2}\,dy$

$=(2y-\frac{1}{6}y^3)\big|_{-2}^{2}$

$=4-\frac{8}{6}-(-4+\frac{8}{6})=\frac{16}{3}$